P1a

(a) The potential due to a point charge ($+Q$) at a distance 'r' away from it is given by

$$V_r = \frac{kQ}{r}$$

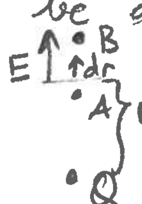
Since B, & C are the same distance away, this would imply that $V_B = V_C = \frac{kQ}{R}$

By this, the difference in potential between A & C should be the same as the difference in potential between A & B.

1b For $A \to B$, we have already seen in class that if we use the definition,

$$V_B - V_A = -\int_A^B (q\vec{E} \cdot d\vec{r})$$

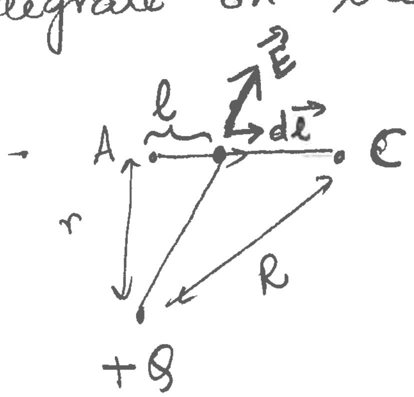
& since $\vec{E}$ and $d\vec{r}$ will be aligned in moving from $A \to B$



$$\vec{E} \cdot d\vec{r} = E\,dr = \frac{kQ}{r^2}\,dr$$

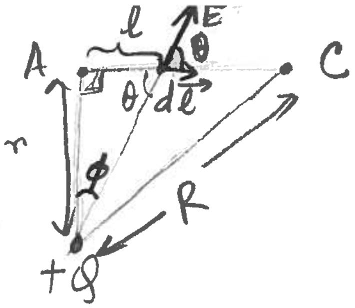
The integration gives $V_B - V_A = kQ\left(\frac{1}{R} - \frac{1}{r}\right)$

But the real question is what happens when we integrate on the path from $A \to C$.



Let me consider a position, l away from A along the path. the $d\vec{l}$ & $\vec{E}$ are shown.

Notice how $d\vec{l}$ & $\vec{E}$ are not aligned



(P2)

So, in moving from l to $l+dl$ the drop in potential is given by

$$\left(V_{l+dl} - V_l\right) = -\vec{E} \cdot d\vec{l}$$

Making sense: $V_{l+dl} - V_l = dV$

$\vec{E} \cdot d\vec{l}$ = Work done on a unit test charge in moving through dl

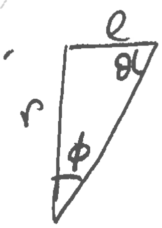
= gain in KE of that unit test charge

$\Rightarrow$ dV & $\vec{E} \cdot d\vec{l}$ must have opposite sign. (gain in KE = loss in PE)

or $dV = -\vec{E} \cdot d\vec{l} = -E\, dl \cos\theta$ (θ = angle between $\vec{E}$ & $d\vec{l}$)

$$= -\frac{kQ}{(r^2+l^2)} dl \cos\theta = -\frac{kQ}{r^2+l^2} dl \sin\phi$$

Note $r, l, \sqrt{r^2+l^2}$ form a right angled Δ.



$\cos\theta = \sin\phi$

Now, $l = r\tan\phi$

$dl = r\sec^2\phi\, d\phi$

& $r^2 + l^2 = r^2\sec^2\phi$

$$\Rightarrow dV = -\frac{kQ}{r^2\sec^2\phi} r\sec^2\phi\, d\phi \sin\phi = -\frac{kQ}{r}\sin\phi\, d\phi$$

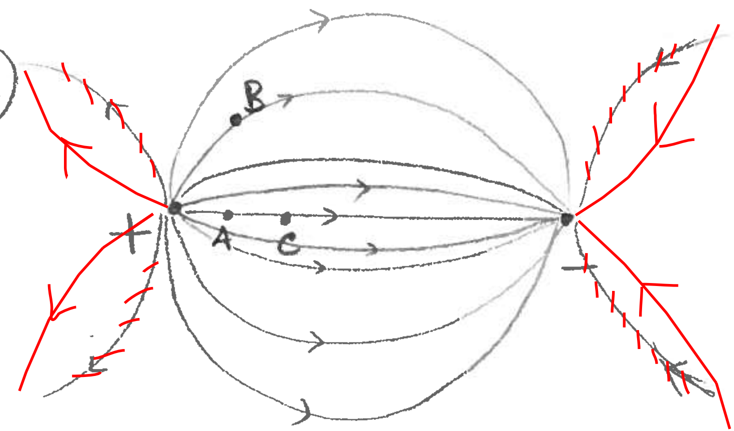
So now integrating over ϕ from $\phi = 0$ to $\phi = \cos^{-1}\left(\frac{r}{R}\right)$ which is the angle ϕ at position C

$$V_C - V_A = \int_A^C dV = -\int_0^{\cos^{-1}(r/R)} \frac{kQ}{r}\sin\phi\, d\phi = +\frac{kQ}{r}\cos\phi\Big|_0^{\cos^{-1}(r/R)}$$

$$= \frac{kQ}{r}\left(\frac{r}{R} - 1\right) = \left(\frac{kQ}{R} - \frac{kQ}{r}\right)$$

$\rightarrow$ which is the same as $V_B - V_A$ & matches my part (a) answer.

2/ (a)



Clearly, the field is strongest at A. Between B & C its trickier. But it would seem that the lines are still denser at C than at B.

So $E_A > E_C > E_B$

magnitudes

(b) Imagine a (+1C) test charge moving from A to C. It would gain speed (repulsion from + & attraction to − both would accelerate it towards C)

So since KE would increase from A → C, so potential energy — and hence potential, since we have a +ve test charge — must decrease

or $V_A > V_C$

Same for A → B. Now, only a part (component) of the field that points up would accelerate the particle, (More like something falling down a ramp rather than falling vertically) so that $V_A > V_B$

what about B & C. → More difficult to ascertain

Suppose B & C are same distance away from ⊕ charge

then $V_B = \underbrace{V_{\text{due to } \oplus}}_{\text{+ve contribution}} + \underbrace{V_{\text{due to } \ominus}}_{\text{−ve contribution}}$ ≈ Since B is further from ⊖, its −ve contribution would be less & so

$V_B > V_C$

→ $V_A > V_B > V_C$ (But the B − C rank depends on exact locations)

2c The particle's initial potential energy converts into kinetic energy, such that total energy is conserved.

$$PE_C + KE_C = PE_A + KE_A$$

Or,
$$qV_C + \frac{1}{2}mv_C^2 = qV_A + \frac{1}{2}mv_A^2$$

We know everything except v_A: we can solve for it

$$v_A = \sqrt{v_C^2 + \frac{2q}{m}(V_C - V_A)} = \sqrt{\left(50\frac{m}{s}\right)^2 + \frac{2(-4\times10^{-3}C)}{(.1\times10^{-3}kg)}\cdot(100-300\,V)}$$

$$\simeq 136 \text{ m/s}$$

⊛ How is it that the particle goes from lower potential ($V_C = 100V$) to higher potential ($V_A = 300V$) & still gained speed from 50 m/s → 136 m/s

↳ True: But since particle is -ve, we have to be careful. Although it moves to the location of higher potential, it loses potential energy because of its -ve charge.

$$PE = qV$$

$$\Rightarrow \Delta PE = q\,\Delta V$$

if q is negative then a positive ΔV would generate a negative ΔPE!

(of course, this makes sense if you think about forces on the particle by the ⊕ & ⊖ charges.

HW 10

$\boxed{\#3}$ (a) As the bead falls, it loses potential energy. Total energy is conserved, and since the bead accelerates, toward the ball, it gains KE, so it must lose P.E.

(b) The bead gains potential. Why?
Potential is defined as the potential energy per unit charge, and since we're talking about a negatively charged bead, if it loses PE, it must gain potential.

Also see the discussion at the end of P2 c

Also, we know that $\Delta V = -\int_{r_i}^{r_f} \vec{E} \cdot d\vec{s}$.
since $\vec{E}$ is pointing outward from the ball, and $d\vec{s}$ is inward, toward the ball, $\vec{E} \cdot d\vec{s}$ is negative, making ΔV positive.

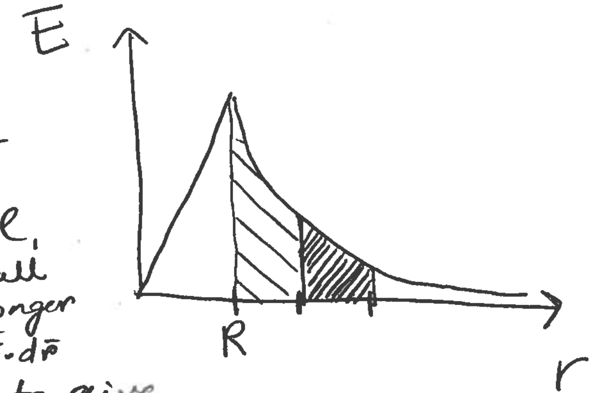
(c) We know that total energy is conserved, so $U_i + K_i = U_f + K_f$. In this case, $K_i = 0$ because the bead is released from rest. The potential energy of the bead is $U = \frac{kQq}{r}$, because the the sphere acts like a point charge. So, we just solve the energy conservation equation for V_f:

This we saw when using Gauss's Law. Outside a sphere of charge the field acts just as if the sphere was a point charge. (But only outside it)

$$\frac{kQq}{r_i} + \cancel{K_i}^{\,0} = \frac{kQq}{r_f} + \frac{1}{2} m v_f^2$$

$$V_f = \sqrt{\frac{2kQq}{m}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)}$$

3(d) The potential change is greater over the second 15 cm of the fall. The E-field is changing more rapidly closer to the sphere, since $E \propto 1/r^2$. To find ΔV, you can integrate the E-field over the distance traveled, and since the distance is the same for both segments, then ΔV must be greater closer to the sphere.



Also $\Delta V = -\int_A^B \vec{E}\cdot d\vec{r}$. So, closer to the ball if the field is stronger

If E is bigger that implies larger $\vec{E}\cdot d\vec{r}$ contributions. So they would add up to give a larger ΔV.

$\boxed{\#4}$ This problem is similar to 3c - again, we know energy is conserved. In this case, the bead is still moving ($v=100$ m/s) very far away from the sphere ($r \to \infty$). We set $V=0$ at $r=\infty$. (Theoretically, we can set $V=0$ anywhere, but this usually simplifies the math!)

$$U_i + K_i = \cancel{U_f}^{\,0} + K_f$$

minus sign taking bead to have charge of $(-q)$

$$-\frac{kQq}{r_i} + \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2$$

$$\Leftarrow v_i = \sqrt{\frac{2}{m}\left(\frac{1}{2}mv_f^2 + \frac{kQq}{r_i}\right)}$$

We know $v_f = 100$ m/s and r_i = radius of the large ball

So we can calculate v_i

or, $$V_i = \sqrt{V_f^2 + \frac{2k}{m}\frac{Qq}{r_i}} \quad (1)$$

Careful when putting in values. We said charge on bead is $(-q)$ so when we put in the value q would be a positive value.

Alternatively you could have written the equations at

$$\frac{kQq}{r_i} + \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2$$

or $$V_i = \sqrt{V_f^2 - \frac{2k}{m}\frac{Qq}{r_i}} \quad (2)$$

Here signs are implicitly included in symbols.

For (1) you put $q = 2\,nC$

for (2) you put $q = -2\,nC$ (because we have left the sign inside of q there)

Either way we should get $V_i > V_f$. Since the bead is attracted to the ball, I must have to throw it with a very high speed so that it is still moving at V_f, even at a large distance away

5. Two long cylindrical rods have the same length, mass and positive charge. The only difference is that rod 1 is made of metal while rod 2 is made of an insulting material through which the charge is spread uniformly. For which rod, if either, is...

(a) the potential greater at the surface of the rod (far from the end)? Explain.

The difference in the two situations is that for the metal rod the charges will all repel until they're all at the surface of the rod. For the insulated rod, the charges have to stay fixed uniformly through the rod.

What difference does this cause in the electric field outside the rod? Well, for both rods, in any slice of the rod you should still have the same amount of total charge. For the metal rod, the charge will be on the outside surface and for the insulted rod, the charge will be uniform, but the total will be the same. So if you do Gauss' Law by drawing equal sized Gaussian cylinders around some piece of the rods, then the charge enclosed and the area with flux going out will be the same. Since in both cases the electric fields will be radially pointing away from the rod, then the Electric fields must be equal (you can confirm it by actually doing Gauss' Law and finding the electric field). The point is that for every point outside the rod, the electric fields of the two rods are the same. So charges released from the surfaces of the rods will experience the same work done as they go out into infinity, so the potentials at the surfaces of the rods are the same.

(b) the potential greater at the center of the rod? Explain.

Inside the rod is a different situation. If you draw a Gaussian cylinder inside the rods, then for the metal rod, the charge inside will be zero; all the charge is on the outside surface of the rod. Since the field through the surface of the Gaussian sphere would have to be perpendicular to the surface, this means that the electric field has to be zero inside the cylinder (try it!).

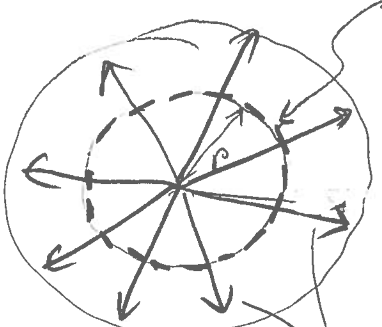
For the insulated rod, there will always be some charge inside any Gaussian cylinder inside the rod, so the electric field in the cylinder won't be zero (except at the center axis).

What does this mean for the potential at the center of the rod? Imagine you start a charge at the center of the rod, give it a small tap (because it won't move on its own if it's at the center), and see how much work the electric field of the rod does as the charge moves to the surface of the rod. In the metal rod, there's no electric field, so no work done. That means for the metal rod that any point inside the rod has the same potential as the surface of the rod. For the insulated rod, the charge will speed up as it approaches the surface of the cylinder due to the electric field, work is being done on the charge. Therefore, in this case the potential at the center is greater than at the surface of the rod.

(c) the electric field greater at a point inside the rod but not on the central axis? Explain.

See part (b)

P6



Gaussian Sphere.

By the symmetry of the problem, here is what the field lines would look like

Spherically symmetric field lines.

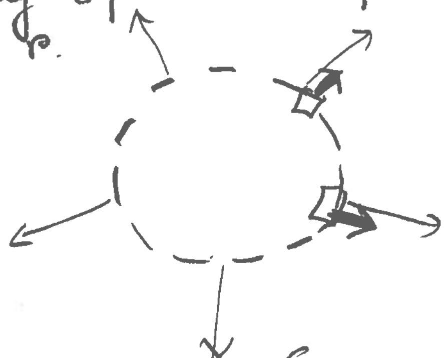
Since the field is spherically symmetric, I would pick a spherical Gaussian Surface. (dotted lines

Remember: Though I'm drawing circles, they are really spheres)

By gauss's law, $\Phi_{\text{sphere of radius } r} = \dfrac{q_{enclosed}}{\epsilon_0}$

Since I'm interested in the field at radius r, I let my sphere be of radius r.

(1) Flux :



The field vector has the same magnitude everywhere on the Gaussian sphere. & the area vector (short, fat arrows) are aligned in same direction as the field vector, locally

E is same at all points on surface

$$\text{So} \quad \Phi = \int_{\text{Surface}} \vec{E} \cdot d\vec{A} = \int_{\text{Surface}} E\, dA = E \int_{\text{Surface}} dA$$

because $\vec{E} \cdot d\vec{A} = E\, dA \cos(0^\circ) = E\, dA$

$$= E\, 4\pi r^2$$

(2) Charge Enclosed : Total charge Q is spread over volume of $\frac{4}{3}\pi R^3$. We want charge within $\frac{4}{3}\pi r^3$

$$q_{enclosed} = \frac{Q}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3 = Q \frac{r^3}{R^3}$$

Now, by Gauss's Law

$$E\, 4\pi r^2 = Q \frac{r^3}{R^3} \cdot \frac{1}{\epsilon_0}$$

or $$\boxed{E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \cdot r}$$ — for $\underbrace{r<R}$

Remember, this is valid only inside the sphere because $q_{enclosed}$ behaves differently if $r>R$ (pts outside)

(b) At the center, we can simply put $r=0$

$$E(r=0) = 0 \quad [\text{from part (a)}]$$

→ But anyways, that is expected. At the center, there are equal pulls in all directions that reduce the field to zero (since $\vec{E}$ is a vector)

(c) V_{center} potential.

- There are 2 ways to calculate

(1) $V_A = \int_A^{\infty} \vec{E} \cdot d\vec{r}$

(2) $V_A = \int \frac{k\, dQ}{r}$
the whole charge distribution

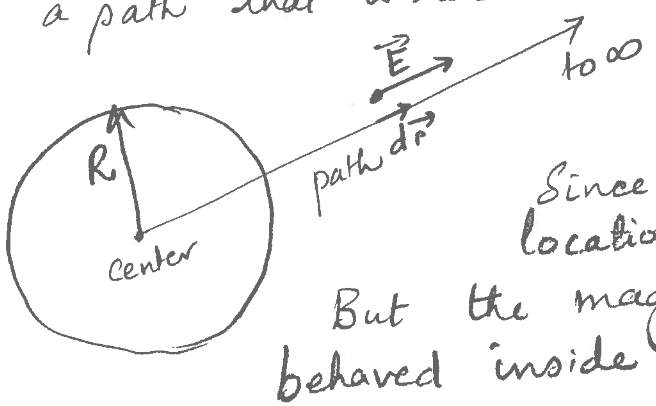
Since I already know the field, I'm going to go with (1) for now.

But wait, $E_{inside}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r$ → part (a)

What if $r>R$. Again using Gauss's Law it's easy to see that $E_{\substack{outside \\ r>R}}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2}$

So $V_{center} = \int_{r=0}^{r=\infty} \vec{E} \cdot d\vec{r}$ This integral is over any path from center to infinity

Since, the field is radial, it might be wiser to choose a path that is radial too → the path & a $d\vec{r}$ vector is shown



The $\vec{E}$ at that same location is also shown.

Since $\vec{E}$ is parallel to $d\vec{r}$ at all locations on this path $\vec{E} \cdot d\vec{r} = E\,dr$

But the magnitude of $\vec{E}$ is differently behaved inside & outside the charged sphere.

$$E_{\substack{inside \\ r<R}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r \qquad E_{\substack{outside \\ r>R}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

(Notice how both functions have the same value at $r = R$)

$$\text{So } V_{center} = \underset{\text{along path}}{\int_{r=0}^{r=\infty} \vec{E} \cdot d\vec{r}} = \int_{r=0}^{r=\infty} E\,dr = \int_{r=0}^{r=R} E\,dr + \int_{r=R}^{r=\infty} E\,dr$$

$$= \int_{r=0}^{R} \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r\,dr + \int_{r=R}^{r=\infty} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dr$$

$$= \frac{Q}{4\pi\epsilon_0 R^3}\left(\frac{R^2}{2} - 0\right) + \frac{Q}{4\pi\epsilon_0}\left[-0 - \left(-\frac{1}{R}\right)\right]$$

$$= \underbrace{\frac{Q}{8\pi\epsilon_0 R}}_{\text{Extra contribution from surface to center}} + \underbrace{\frac{Q}{4\pi\epsilon_0 R}}_{\text{potential at surface}}$$

[Remember $\underset{\text{constant in Coulomb's Law}}{k} = \frac{1}{4\pi\epsilon_0}$]

$$V_{center} = \frac{3}{2} \frac{Q}{4\pi\epsilon_0 R}$$

Making Sense? Well we should expect potential to be more than $\frac{kQ}{R}$ ($=V_{surface}$). Because there is a radially outward field, a positive test charge would gain KE as it moves from center towards the outer edge of charged sphere. If it gains KE, it must be losing PE.

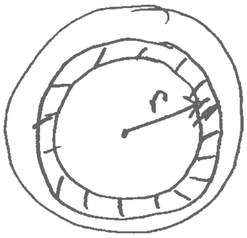
$\Rightarrow$ As we go towards center, the locations have greater potential.

If R increases (for same Q) more bits of charge are further away and so the contribution to potential should be less (Think of it also as the acceleration & hence the gain in KE would be less if charge bits are further away) So it makes sense that $V_{center} \propto \frac{1}{R}$

More charge $\rightarrow$ ofcourse, more V in this case. So it makes sense that $V \propto Q$

So $V = \frac{3}{2}\frac{kQ}{R} = \frac{3}{2}\frac{Q}{4\pi\epsilon_0 R}$ makes sense.

6e Alternatively, I can try & break the sphere of charge in expanding shells of charge



Shaded is a spherical shell of radius 'r' & thickness dr.

For each little bit of charge dQ on the shell the contribution to potential is $\frac{k\,dQ}{r}$. Since every bit's contribution is same, the total contribution by the shell

is $$dV_{\text{shell, center}} = \frac{k(dQ_{\text{shell}})}{r}$$

How much is dQ_{shell}? $$dQ_{\text{shell}} = \frac{Q}{\text{Total Volume of sphere}} \cdot \left(\text{Volume of shell}\right)$$

$$= \frac{Q}{\frac{4}{3}\pi R^3} \cdot \left(4\pi r^2 dr\right)$$

$$= \frac{3Q}{R^3} r^2 dr$$

or, $$dV_{\text{shell, center}} = \frac{3kQ}{R^3} r\,dr$$

To get the total potential we integrate over all shells.

or $$V_{\text{center}} = \int_0^R \frac{3kQ}{R^3} r\,dr = \frac{3kQ}{R^3}\left[\frac{r^2}{2}\right]_0^R = \frac{3}{2}\frac{kQ}{R}$$

Note: this is the same answer as the other $\int \vec{E}\cdot d\vec{r}$ method since $k = \frac{1}{4\pi\epsilon_0}$!

(6d) What about $E = 0$ at center!

Well $\Delta V = -\int \vec{E} \cdot \vec{dr}$ As $E \to 0$ it just means that there are smaller and smaller contributions to the change in potential as we move towards the center.

So true that as we move towards center $E \to 0$ but that just means that the change in potential tends to zero.; not the value of the potential itself.

This is the classic question of is velocity $= 0$ when acceleration $= 0$! No, $a = 0$ implies that velocity is constant, but it can be a non-zero constant. Similarly for potential, $E \to 0$ just means that potential isn't changing much and right at the center, the change in potential $= 0$, (But the value of potential is non-zero) $\to$ no contradiction there.

Alternatively, $E = -\frac{dV}{dr}$ so $E = 0 \Rightarrow$ the slope of potential graph is zero. But slope $= 0$ does not mean that the variable value is zero